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Consistent Histories Approach to the Unruh Effect

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Using the history projection operator (HPO) approach to consistent histories we rederive Unruh's result that an observer constantly accelerating through the Minkowski vacuum appears to be immersed in a thermal bath. We show that propositions about any symmetry of a system always form a consistent set and that the probabilities associated with such propositions are decided by their value in the initial state. We use this fact to postulate a condition on the decoherence functional in the HPO setup. Finally we show that the Unruh effect arises from the fact that the *initial* density matrix corresponding to the inertial vacuum can be written as a thermal density matrix in the Fock basis associated with the accelerating observer.

1. INTRODUCTION

1.1. Consistent Histories

The consistent histories approach to quantum theory originated in the pioneering work of Griffiths (1984) and Omnes (1988). Initially the formalism was developed in an attempt to escape the familiar difficulties of the Copenhagen interpretation. More recently, Gell-Mann and Hartle (1990) suggested that generalised history theories may be useful in tackling the problems of quantum cosmology and quantum gravity, in particular the problem of time.

The basic ingredient of "conventional" consistent histories is a time-ordered sequence of propositions about the system represented by a class operator:

$$C_\alpha := \alpha_{t_1}(t_1)\alpha_{t_2}(t_2)\cdots\alpha_{t_n}(t_n) \tag{1}$$

where $\alpha_{t_i}(t_i)$ is a Heisenberg picture projection operator representing a proposition made about the system at time t_i. To make physical predictions we must use the decoherence functional to identify (strongly) consistent sets of histories, that is,

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sets $\{\alpha_i\}$ such that,

$$d(\alpha_i, \alpha_j) := \mathrm{Tr}_{\mathcal{H}}[C^{\dagger}_{\alpha_i} \rho C_{\alpha_j}] \tag{2}$$

$$= 0 \quad \text{if } i \neq j \tag{3}$$

Within such consistent sets, the probability of a particular history α_i "occuring" is $d(\alpha_i, \alpha_i)$. The consistency condition guarantees that the Kolmolgorov sum rules are satisfied.

If generalised history theories are to be useful in formulating quantum gravity, then it is important to understand how more conventional theories such as nonrelativistic quantum mechanics and quantum field theory (QFT) can be formulated in history language. While nonrelativistic quantum mechanics has been extensively studied within the formalism, there are very few results concerning QFT. This is the motivation for this paper in which we rederive a well-known result in the theory of QFT on curved spaces, from a histories perspective. The Unruh effect (1976) is an analogue of Hawking radiation, but the gravitational field that induces the radiation is "apparent" rather than "real," That is, it is measured by an observer accelerating through empty space rather than by an observer in the gravitational field of a black hole.

1.2. The HPO Approach

Isham (1994) proposed an algebraic scheme for generalised history theories of the type suggested by Gell-Mann and Hartle. The algebraic axioms are set up in analogy with the logical approach to single-time quantum theory, which is concerned with the pair $(\mathcal{L}, \mathcal{S})$ where $\mathcal{L}$ is the lattice of projection operators on a Hilbert space and $\mathcal{S}$ is the set of density matrices. Isham proposed that a generalised history theory should be composed of the pair $(\mathcal{UP}, \mathcal{D})$ where $\mathcal{UP}$ is an *orthoalgebra* of propositions about possible histories and $\mathcal{D}$ is the space of decoherence functionals.

To fit conventional consistent histories into these axioms, we would like to interpret the class operators as logical propositions; however, the product of noncommuting projection operators is not a projection operator. This means it is difficult to define conjunctions, disjunctions, and negations consistently. However, the *tensor product* of two projectors *is* a projector on the tensor product space. This is the central idea of the history projection operator (HPO) approach to consistent histories. The tensor product of Schrödinger picture projection operators, $\alpha_{t_1} \otimes \alpha_{t_2} \otimes \cdots \otimes \alpha_{t_n}$, which is a projector on the n-time history space, $\mathcal{V}^n := \mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2} \otimes \cdots \otimes \mathcal{H}_{t_n}$, represents the proposition "α_{t_1} is true at time t_1 *and then* α_{t_2} is true at time t_2 ... *and then* α_{t_n} is true at time t_n." Now we can define the logical operations as we would for projection operators in any Hilbert space. So in this case, the orthoalgebra $\mathcal{UP}$ is in fact the lattice $\mathcal{P}(\mathcal{V}^n)$ of projection operators on the history space.

The decoherence functional (2) can be written as

$$d(\alpha_i, \alpha_j) = \mathrm{Tr}_{\mathcal{V}^n \otimes \mathcal{V}^n}(\alpha_i \otimes \alpha_j X) \tag{4}$$

for some $X \in \mathcal{B}(\mathcal{V}^n \otimes \mathcal{V}^n)$ where $\mathcal{B}(\mathcal{H})$ is defined as the set of bounded operators on $\mathcal{H}$. Conversely Gleason's theorem can be used to show that any decoherence functional that satisfies certain natural conditions can be written in this form (Isham *et al.*, 1994). Therefore $\mathcal{D}$, the space of decoherence functionals, is the set of all functionals of this form. This result also holds in the continuous-time case (Isham and Linden, 1995).

2. THE SIMPLE HARMONIC OSCILLATOR

2.1. Continuous Times

In extending HPO theory to the case of continuous time, which we anticipate to be important for QFT, we encounter the continuous tensor product of the single-time Hilbert space: $\mathcal{V}^{\mathrm{cts}} := \otimes_{t\in\mathbb{R}}\mathcal{H}_t$. To deal with this object it is useful to confine ourselves for the moment to the simple harmonic oscillator (SHO) (where $\mathcal{H}_t = L^2(\mathbb{R})$) and to consider the *history group* (Isham and Linden, 1995). We can view $\mathcal{V}^n$ arising as the representation space for the n-fold direct product of the Weyl group of single-time quantum theory on the line

$$[x_{t_i}, x_{t_j}] = 0 \tag{5}$$

$$[p_{t_i}, p_{t_j}] = 0 \tag{6}$$

$$[x_{t_i}, p_{t_j}] = i\hbar\delta_{ij} \tag{7}$$

The advantage of this perspective is that it can be readily generalised to the case of continuous time. For this we consider the algebra

$$[x_f, x_g] = 0 \tag{8}$$

$$[p_f, p_g] = 0 \tag{9}$$

$$[x_f, p_g] = i\hbar(f, g) \tag{10}$$

where $f, g \in L^2(\mathbb{R})$; $x_f := \int dt\ f(t)x_t$; and $(f, g) := \int dt\ f(t)g(t)$. This algebra is clearly isomorphic to the algebra of a one-dimensional QFT and suggests that field theory techniques will be useful in studying the theory. It is well-known that this algebra has a representation on the Fock space over $L^2(\mathbb{R})$, denoted $\mathcal{F}[L^2(\mathbb{R})]$. Indeed it can be shown that (Isham and Linden, 1995),

$$\mathcal{V}^{\mathrm{cts}} := \otimes_{t\in\mathbb{R}}\mathcal{H}_t \sim \mathcal{F}[L^2(\mathbb{R})] \tag{11}$$

and again $\mathcal{UP}$ is a lattice, now it is the set of projection operators on the continuous-history space, $\mathcal{P}(\mathcal{V}^{\mathrm{cts}})$. The condition that the time-averaged Hamiltonian is self-adjoint is sufficient to select a unique representation of the history algebra (Isham and Linden, 1995). This representation is defined by the Fock basis associated with the creation operator,

$$a_f^\dagger := \sqrt{\frac{m\omega}{2\hbar}}x_f - i\sqrt{\frac{1}{2m\omega\hbar}}p_f \tag{12}$$

2.2. Time–Averaged Propositions

The physical interpretation of a continuous-time HPO theory is based on the assumption that projectors onto the spectrum of self-adjoint operators on $\mathcal{V}$ represent propositions about the time-averages of physical quantities. So projections onto the eigenvectors of the x_f operators introduced in (8) represent propositions about the average position of the particle over time. As $[x_f, x_g] = 0$, these operators have common eigenvectors for any smearing function. We denote these eigenvectors $|x(\cdot)\rangle$ and they can be interpreted as fine-grained histories or trajectories of the particle. In the single-time theory, $x_t|x\rangle = x|x\rangle$. So, formally, $x_t|x(\cdot)\rangle = x(t)|x(\cdot)\rangle$, which suggests,

$$x_f|x(\cdot)\rangle := \int dt\ f(t)x_t|x(\cdot)\rangle = (f, x)|x(\cdot)\rangle \tag{13}$$

If this is to make sense then $x(\cdot)$ must be a member of $L^2(\mathbb{R})$. However, it is likely that the eigenvectors $x(\cdot)$ will be distributions rather than functions. The natural procedure now would be to interpret the symbol (f, x) to be the real number obtained from the pairing of the distribution x with the function f. This implies that the allowed functions f should really be members of Schwartz space rather than $L^2(\mathbb{R})$. We will not confront this issue here, and just consider functions that are members of some unspecified space, τ.

For each $f \in \tau$ we have an equivalence relation, $\sim_f$, on trajectories if we define $x(\cdot) \sim_f y(\cdot)$ if $(f, x) = (f, y)$. We denote these equivalence classes by $[(f, x)]$. Now we consider projections onto the spectrum of x_f. We denote the operator that projects onto the eigenvector of x_f with eigenvalue (f, x) as $P_{(f,x)}$; it projects onto the equivalence class of trajectories $[(f, x)]$, that is, onto a coarse-grained history. Similar remarks obviously apply to operators $P_{(f,p)}$, which project onto coarse-grained momentum trajectories $[(f, p)]$.

Another operator of physical significance is the smeared Hamiltonian:

$$H_f := \int dt\ f(t)\left(\frac{1}{2m}p_t p_t + \frac{m\omega^2}{2}x_t x_t\right) \tag{14}$$

$$= \hbar\omega \int dt\ f(t)\left((a_t^\dagger a_t) + \frac{1}{2}\right) \tag{15}$$

Projections onto its spectrum represent propositions about the time-averaged energy of the system.

For our purposes, the average number operator N will be of prime importance. We can formally define it as follows:

$$N := \int dt\, a_t^\dagger a_t \tag{16}$$

The eigenvectors of this operator are vectors of the form

$$|n_f\rangle := (n!)^{-1/2} \int dt\;\, f(t)(a_t^\dagger)^n\, |0\rangle \tag{17}$$

These are also eigenvectors of the Hamiltonian. The average number operator has a highly degenerate spectrum as vectors of the above form have eigenvalue $n \in \mathbb{N}$ for all smearing functions f, as can be easily checked. We will denote the projection operator onto $|n_f\rangle$ as P_{n_f}; it represents a proposition about the average number of quanta present in a particular time interval.

2.3. Propositions within a Finite Time Interval

We can write the average number operator defined here in the form $N = N_{f=1}$, where $N_f := \int dt\;\, f(t) a_t^\dagger a_t$. This shows that there is a problem with the definition because the constant function $f = 1$ is not a member of $L^2(\mathbb{R})$. However it is a member of $L^2[a, b]$, where $[a, b]$ is a finite interval of the real line. This suggests that we should really be dealing with propositions in a finite interval of time.

Consider again the proposition P_{n_f}. Intuitively the support of f affects the time period in which the proposition is made. In other words if $supp(f) \subset [a, b]$ then the proposition P_{n_f} refers to the average number of particles during the time period $[a, b]$. We can formulate this rigorously by splitting up $\mathcal{V}^{\text{cts}}$ as follows:

$$\mathcal{V}^{\text{cts}} := \otimes_{t\in\mathbb{R}} \mathcal{H}_t \tag{18}$$

$$= \mathcal{V}^{[-\infty,a]} \otimes \mathcal{V}^{[a,b]} \otimes \mathcal{V}^{[b,\infty]} \tag{19}$$

where $\mathcal{V}^{[a,b]} := \otimes_{t\in[a,b]} \mathcal{H}_t$. Now we can use the isomorphisms

$$\otimes_{t\in[a,b]}\, e^{L_t^2[a,b]} \sim e^{\otimes\int_a^b L_t^2[a,b]} \sim \mathcal{F}[L^2[a, b]] \tag{20}$$

(Guichardet, 1972). Here, ${}^{\otimes}\!\int_a^b L_t^2[a, b]$ is the direct integral Hilbert space over the interval $[a, b]$. An element of this Hilbert space, F, can be considered as a one-parameter family of elements of $L^2[a, b]$, which we denote by f_t, where $t \in [a, b]$. The inner product is defined as

$$(F, G)_{\otimes\int_a^b L_t^2[a,b]} := \int_a^b dt (f_t, g_t)_{L_t^2[a,b]} \tag{21}$$

From the right hand side of (20) we can see that $\mathcal{V}^{[a,b]}$ naturally carries a representation of the Lie algebra

$$[x_f, x_g] = 0 \tag{22}$$

$$[p_f, p_g] = 0 \tag{23}$$

$$[x_f, p_g] = i(f, g) \tag{24}$$

where $f, g \in L^2[a, b]$. The natural interpretation of these operators is that they are associated with time-averaged propositions about position and momentum in the finite time interval $[a, b]$. We can form complex combinations of these operators in the usual way to define creation and annihilation operators. Projections onto the eigenvectors of the average number operator associated with these correspond to propositions about the average number of particles in the time interval $[a, b]$.

We can now see that propositions on $\mathcal{V}^{\text{cts}}$ smeared by functions in a finite time interval are isomorphic with propositions on $\mathcal{V}^{[a,b]}$ by

$$P_{nf} \sim \mathbb{I}_{\mathcal{V}[-\infty,a]} \otimes P^{[a,b]}_{n_f} \otimes \mathbb{I}_{\mathcal{V}[b,\infty]} \tag{25}$$

where $f \in L^2[a, b]$ and $P^{[a,b]}_{n_f} \in \mathcal{P}(\mathcal{V}^{[a,b]})$. So from now on when we use the average number operator N it should be understood that in fact we are averaging over a finite interval, that is, we are smearing with functions $f \in L^2[a, b]$.

This is consistent with the definition of finite time interval projectors for coherent states given by Isham and Linden (1995).

2.4. The Decoherence Functional

Isham and Linden (1995) and Anastopolous (2000) have defined decoherence functionals for continuous-time projectors in the HPO scheme by considering projections onto coherent states. However, we are interested in propositions concerning the average number of quanta. These cannot be simply related to coherent states, so we will take a different approach and require our decoherence functional to respect the dynamical time translation symmetry of quantum theory. As the projectors onto eigenstates of N commute with the Hamiltonian, we would expect the probability of any such proposition to be decided by its probability in the initial state. We will see that this is indeed the case and that these propositions also form a "canonical" consistent set. We shall then require these conditions to hold in the HPO formalism to obtain a condition on the decoherence functional. Analogous remarks apply to any symmetry of the system, that is, propositions regarding the spectral projectors of any operator that commutes with the Hamiltonian will form a consistent set and their probabilities will be decided by their value in the initial state.

Let us first examine the discrete time case for the SHO with single-time number operator defined by $N^{\text{st}} := a^\dagger a$. Here we have time translation symmetry

$[H, N^{\text{st}}] = 0$, which corresponds to the conservation of the number of quanta. We begin by considering a 2-time history in the conventional setup. It has eigenvectors $|n\rangle := (n!)^{-1/2}(a^\dagger)^n|0\rangle$ and we denote Schrödinger picture projectors onto these vectors by P_n. The class operator takes a particularly simple form, $C_{n_1 n_2} := P_{n_1}(t_1) P_{n_2}(t_2) = P_{n_1} P_{n_2} = \delta_{n_1 n_2} P_{n_1}$. The decoherence functional is then,

$$d_{\text{SHO}}(m_1 m_2, n_1 n_2) := \text{Tr}_{\mathcal{H}}[C_{m_1 m_2} \rho C^\dagger_{n_1 n_2}] \tag{26}$$

$$= \delta_{m_1 m_2} \delta_{n_1 n_2} \text{Tr}_{\mathcal{H}}[P_{m_1} \rho P_{n_1}] \tag{27}$$

$$= \delta_{m_1 m_2} \delta_{n_1 n_2} \delta_{m_1 n_1} \rho_{m_1 n_1} \tag{28}$$

We can see that the fact that the projectors commute with the Hamiltonian means that they must all project onto the same state for the answer to be nonzero. This shows that propositions about the average number of particles, or more generally propositions about any symmetry of a system, always make up a consistent set. It is also clear that the probabilities assigned to these propositions depend on the initial state alone.

Now we examine this in the HPO scheme. The history space is $\mathcal{V}^2 = \mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2}$. We can write the above decoherence functional as a trace over $\mathcal{H}^{\otimes 5} := \mathcal{H}_{t_0} \otimes \mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2} \otimes \mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2}$ using the trick in (Isham and Linden, 1995):

$$d_{\text{SHO}}(m_1 m_2, n_1 n_2) = \text{Tr}_{\mathcal{H}^{\otimes 5}}[\rho \otimes P_{m_1} \otimes P_{m_2} \otimes P_{n_1} \otimes P_{n_2} S_5] \tag{29}$$

Tracing over the initial Hilbert space we obtain,

$$d_{\text{SHO}}(m_1 m_2, n_1 n_2) = \text{Tr}_{\mathcal{V}^2 \otimes \mathcal{V}^2}[P_{m_1} \otimes P_{m_2} \otimes P_{n_1} \otimes P_{n_2} Z] \tag{30}$$

where $Z \in \mathcal{B}(\mathcal{V}^2 \otimes \mathcal{V}^2)$ and is defined in terms of its matrix elements in the energy basis as

$$\langle i_1 \cdots i_4 | Z | j_i \cdots j_4 \rangle = \delta_{i_1 j_2} \delta_{i_2 j_3} \delta_{i_3 j_4} \rho_{i_4 j_1} \tag{31}$$

Now it is the operator Z that contains the initial conditions and forces all the projectors to project onto the same state. In fact, by using these energy eigenstates we have removed the dynamics from the decoherence functional and are left only with the initial conditions and temporal structure encoded in the operator Z. Note that this does not uniquely define Z as any Z' defined by $Z' = U^\dagger Z U$ where U is of the form $e^{if(H)} \otimes e^{ig(H)}$ has the same matrix elements if H is the time-averaged Hamiltonian; $H := \int dt\, H_t$.

Consider a continuous-time energy proposition in standard history theory, represented by the class operator $C_{\{n_t\}} := \Pi_t P_{n_t}(t)$. Heuristically, this is going to be zero unless all of the n_t are equal. If they are all equal, to n say, then the infinite product will equal P_n. In this case the decoherence functional will give the same result as before:

$$d_{\text{SHO}}(\{m_s\}, \{n_t\}) = \delta_{mn}\rho_{mn} \quad \text{if } m_s = m\forall s, \ \ n_t = n\forall t \tag{32}$$

$$= 0 \ \text{ otherwise} \tag{33}$$

We can now understand the degeneracy in the spectrum of the average number operator in the HPO approach. It corresponds to the fact that the number of quanta is conserved and must be an integer. Therefore the time-averaged number of quanta must be an integer over any time period.

From the above discussion we require that the continuous-time HPO decoherence functional satisfies

$$d^{\text{cts}}_{\text{SHO}}(n_f, m_g) = \delta_{mn}\rho_{mn} \tag{34}$$

for all functions f, g. This guarantees that

1. The functional $d^{\text{cts}}_{\text{SHO}}$ assigns the correct probabilities to average number propositions. P_{n_f} corresponds to the proposition "There are an average of n quanta over the time interval $t \in supp(f)$." However, we know that the number of quanta is constant in time so the smearing function is irrelevant and that the probability of finding n particles at any time is simply ρ_{nn}.
2. Number propositions still form a consistent set.

There is a class of operators $Z^{\text{cts}} \in \mathcal{B}(\mathcal{V}^{\text{cts}} \otimes \mathcal{V}^{\text{cts}})$ such that the decoherence functional can be written in the form

$$d^{\text{cts}}_{\text{SHO}}(n_f, m_g) := \text{Tr}_{\mathcal{V}^{\text{cts}}\otimes\mathcal{V}^{\text{cts}}}[P_{n_f} \otimes P_{m_g} Z^{\text{cts}}] \tag{35}$$

such Z^{cts} must satisfy,

$$\langle m_f n_g | Z^{\text{cts}} | m'_{f'} n'_{g'} \rangle := \delta_{mn}\delta_{nm'}\delta_{m'n'}\rho_{mn} \tag{36}$$

for all functions f, f', g, g' as can be easily shown by taking the trace over energy eigenstates:

$$\text{Tr}_{\mathcal{V}^{\text{cts}}\otimes\mathcal{V}^{\text{cts}}}[X] = \int \mathcal{D}\mu[m_f]\mathcal{D}\mu[n_g]\langle m_f n_g | X | m_f n_g \rangle \tag{37}$$

The measure $\mathcal{D}\mu[m_f]$ can be assumed to exist because there is a well-defined measure on $\mathcal{V}^{\text{cts}}$ defined in terms of coherent states (Isham and Linden, 1995). The condition (36) only defines Z^{cts} up to a unitary transformation.

3. QUANTUM FIELD THEORY

3.1. The HPO Approach to QFT

We use throughout the signature $(+, - - -)$. To construct an HPO version of canonical QFT on Minkowski space–time, $\mathcal{M}$, we must first foliate $\mathcal{M}$ with a one-parameter family of spacelike surfaces using some timelike vector n^μ, normalised by $\eta_{\mu\nu} n^\mu n^\nu = 1$. Note that this corresponds to a choice of time direction as seen by some inertial observer. This choice obviously breaks Lorentz covariance and an important unsolved problem in the HPO programme is to show the equivalence of theories based on all such slicings. See Isham *et al.* (1998) for a relevant discussion. In this paper however, we will not consider this problem and will just consider slices orthogonal to the vector $n := \partial_{x^0}$ where x^μ is the coordinate system on $\mathcal{M}$ in which our inertial observer is at rest. Now we consider a canonical three-dimensional QFT to be defined on each Cauchy surface $\mathcal{C}_t$, where $\mathcal{C}_t$ is defined by

$$\mathcal{C}_t := \{m \in \mathcal{M} \mid x^0(m) = t\} \tag{38}$$

$\mathcal{M}$ is a globally hyperbolic space–time so these Cauchy surfaces are all isomorphic. In fact they are all homeomorphic to $\mathbb{R}^3$ so $\mathcal{H}_t = \mathcal{F}[L^2(\mathbb{R}^3, d^3x)]$ for all times t. We define the history algebra to be (in nonrigorous unsmeared form),

$$[\phi_{t_1}(\mathbf{x}_1), \phi_{t_2}(\mathbf{x}_2)] = 0 \tag{39}$$

$$[\pi_{t_1}(\mathbf{x}_1), \pi_{t_2}(\mathbf{x}_2)] = 0 \tag{40}$$

$$[\phi_{t_1}(\mathbf{x}_1), \pi_{t_2}(\mathbf{x}_2)] = i\hbar\delta(t_1 - t_2)\delta^3(\mathbf{x}_1 - \mathbf{x}_2) \tag{41}$$

with $x_1 \in \mathcal{C}_{t_1}$. As shown in (Isham *et al.*, 1998), the requirement that the Hamiltonian is self-adjoint is sufficient to select a representation of this algebra on the history space,

$$\mathcal{V}^{\mathcal{M}} := \otimes_{t\in\mathbb{R}} \mathcal{H}_t \sim \mathcal{F}[L^2(\mathcal{M}), d^4x)] \tag{42}$$

This representation is defined by the annihilation operator

$$a_t(\mathbf{x}) := \frac{1}{\sqrt{2}}\left(K_{\mathrm{M}}^{\frac{1}{4}}\,\phi_t(\mathbf{x}) + i\,K_{\mathrm{M}}^{-\frac{1}{4}}\,\pi_t(\mathbf{x})\right) \tag{43}$$

where K_{M} is defined by $(K_{\mathrm{M}}\, f)(t, \mathbf{x}) := (-\nabla_x^2 + m^2)\, f(t, \mathbf{x})$. Equation (43) is a familiar equation in an unusual form. If we write $\phi_t(\mathbf{x})$ in terms of $a_t(\mathbf{k})$ and $a_t^\dagger(\mathbf{k})$ (defined as the three-dimensional Fourier transforms of $a_t(\mathbf{x})$ and $a_t^\dagger(\mathbf{x})$ respectively) then we have

$$\phi_t(\mathbf{x}) = \int \frac{d^3k}{(2\omega_k)^{1/2}}\left(e^{i\mathbf{k}\cdot\mathbf{x}} a_t^\dagger(\mathbf{k}) + e^{-i\mathbf{k}\cdot\mathbf{x}} a_t(\mathbf{k})\right) \tag{44}$$

However, we must not let the familiar form of these equations make us forget that we are dealing with a history theory. In particular we must remember that the $\phi_t(\mathbf{x})$ operator is in the *Schrödinger* picture and the t label that it carries represents the time that a particular proposition is made, that is, it is a *logical* time quite separate from *dynamical* time. We can introduce dynamical time by using a one-parameter unitary group as usual, but this must involve the introduction of a second time label (Savvidou 1999):

$$\phi_t(s, \mathbf{x}) := e^{isH}\phi_t(\mathbf{x})\, e^{-isH} \tag{45}$$

$$= \int \frac{d^3k}{(2\omega_k)^{1/2}}\left(e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_k s)}a_t^\dagger(\mathbf{k}) + e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega_k s)}a_t(\mathbf{k})\right) \tag{46}$$

where $H := \int dt\, H_t \in \mathcal{B}(\mathcal{V}^{\mathcal{M}})$ is the time-averaged Hamiltonian. Another difference with the canonical theory is that only projection operators have any meaning. Here we will be interested in propositions about the number of particles in a particular mode so we now define these:

$$N_{\mathbf{k}} := \int dt\; a_t^\dagger(\mathbf{k})a_t(\mathbf{k}) \tag{47}$$

This operator has a highly degenerate spectrum as vectors of the form

$$\left|n_f^{\mathbf{k}}\right\rangle := (n!)^{-\frac{1}{2}} \int dt\;\; f(t)(a_t^\dagger(\mathbf{k}))^n|0^M\rangle \tag{48}$$

are eigenvectors, with eigenvalue $n \in \mathbb{N}$ for all functions f. This degeneracy is the result of the fact that we are considering a free theory, so each $N_{\mathbf{k}}$ is separately conserved ($[N_{\mathbf{k}}, H] = 0$) and must be an integer. Projectors $P_{n_f^{\mathbf{k}}}$, which project onto these vectors, represent propositions about the average number of particles in mode $\mathbf{k}$ in the interval $t \in \text{supp}(f)$. Symmetry implies that the propositions $P_{n_f^{\mathbf{k}}}$ form a canonical consistent set and that the probability of these propositions is decided by the probability in the initial state:

$$d^{\mathcal{M}}\left(m_f^{\mathbf{k}}, n_g^{\mathbf{k}'}\right) = \delta_{mn}\delta^3(\mathbf{k}-\mathbf{k}')\rho^M_{m^{\mathbf{k}}n^{\mathbf{k}'}} \tag{49}$$

for all f, g, where $\rho^M \in \mathcal{B}(\mathcal{H}_{t_0})$ is defined by its matrix elements:

$$\rho^M_{m^{\mathbf{k}}n^{\mathbf{k}}} := \langle m^{\mathbf{k}}|\rho^M|n^{\mathbf{k}}\rangle \tag{50}$$

and $|n^{\mathbf{k}}\rangle := (a_{t_0}^\dagger(\mathbf{k}))^n|0^M\rangle$.

We can write the decoherence functional in the form

$$d^{\mathcal{M}}\left(m_f^{\mathbf{k}}, n_g^{\mathbf{k}'}\right) = \text{Tr}_{\mathcal{V}^{\mathcal{M}}\otimes\mathcal{V}^{\mathcal{M}}}\left[P_{m_f^{\mathbf{k}}} \otimes P_{n_g^{\mathbf{k}'}}\mathcal{Z}^{\mathcal{M}}\right] \tag{51}$$

if $\mathcal{Z}^{\mathcal{M}} \in \mathcal{B}(\mathcal{V}^{\mathcal{M}} \otimes \mathcal{V}^{\mathcal{M}})$ satisfies

$$\left\langle m_f^{k_1} n_g^{k_2}\middle|\mathcal{Z}^{\mathcal{M}}\middle|m'^{k_1'}_{f'} n'^{k_2'}_{g'}\right\rangle = \delta_{mn}\delta_{nm'}\delta_{m'n'}\,\delta(\mathbf{k}_1-\mathbf{k}_2)\,\delta(\mathbf{k}_2-\mathbf{k}_1')\,\delta(\mathbf{k}_1'-\mathbf{k}_2')\,\rho_{m^{\mathbf{k}_1}n^{\mathbf{k}_1}} \tag{52}$$

for all f, f', g, g', which only defines $\mathcal{Z}^{\mathcal{M}}$ up to a unitary transformation as before.

3.2. Canonical QFT on Rindler Space–Time

Consider an observer accelerating with constant acceleration, α, through $\mathcal{M}$. Let ξ^μ denote the coordinates in which this observer is at rest. Then ξ^μ are related to the coordinates x^μ by

$$(x^1)^2 - (x^0)^2 = (\xi^1)^2, \qquad x^0/x^1 = \tanh(\alpha\xi^0),$$
$$x^2 = \xi^2, \qquad x^3 = \xi^3 \tag{53}$$

So, constantly accelerating observers follow hyperbolae in $\mathcal{M}$. These hyperbolae split into two sets depending on the sign of ξ^1. Rindler space, $\mathcal{R}$, is defined to be the space covered by the coordinates ξ^μ with $\xi^1 > 0$. It corresponds to the wedge $x > |t|$ in ordinary Minkowski coordinates. Similarly, $\mathcal{L}$ is defined to be the space covered by ξ^μ with $\xi^1 < 0$. It corresponds to the the wedge $x < |-t|$. The metric in these coordinates takes the form,

$$ds^2 := g_{\mu\nu}\, d\xi^\mu\, d\xi^\nu = (\alpha\xi^1)^2(d\xi^0)^2 - (d\xi^1)^2 - (d\xi^2)^2 - (d\xi^3)^2 \tag{54}$$

The vector ∂_{ξ^0} is a globally timelike Killing vector field in $\mathcal{R}$. Therefore $\mathcal{R}$ is globally hyperbolic and we can formulate QFT canonically by using ∂_{ξ^0} to select a particular representation of the canonical commutation relations. On nonglobally hyperbolic space–times there is no globally timelike vector field and therefore no way to select one of the infinite number of unitarily inequivalent representations. This is the major difficulty in the theory of QFT in curved spaces. However, this does not concern us here and we proceed by solving the classical Klein–Gordon equation in curved space–time:

$$(g^{\mu\nu}\nabla_\mu\nabla_\nu + m^2)\,\phi^R(\xi) = 0 \tag{55}$$

Here, ∇_μ is the covariant derivative associated with the metric (54). As shown in Fulling (1973), Eq. (55) can be reduced to a Bessel equation with solutions $u^R_\kappa(\xi)$. Following the canonical procedure we now second quantise and expand the quantum field in terms of creation and annihilation operators,

$$\phi^R(\xi) := \int \frac{d^3\kappa}{(2\omega_\kappa)^{1/2}} \left(u^R_\kappa(\xi)\, b^R(\vec{\kappa})^\dagger + u^R_\kappa(\xi)\, b^R(\vec{\kappa})\right) \tag{56}$$

We can write down a similar equation for the field in $\mathcal{L}$ and because $\mathcal{C}^{\mathcal{L}}_\tau \cup \mathcal{C}^{\mathcal{R}}_\tau$ is a Cauchy surface for $\mathcal{M}$ we can expand the field on $\mathcal{M}$ as

$$\phi(x) = \int \frac{d^3\kappa}{(2\omega_\kappa)^{1/2}} \big(b^R(\vec{\kappa})\, \bar{u}^R_\kappa(x) + b^R(\vec{\kappa})^\dagger\, \bar{u}^R_\kappa(x)$$
$$+ b^L(\vec{\kappa})\, \bar{u}^L_\kappa(x) + b^L(\vec{\kappa})^\dagger\, \bar{u}^L_\kappa(x)\big)$$

where

$$\bar{u}^R_\kappa(x) := u^R_\kappa(x) \quad \text{if } x \in \mathcal{R} \tag{57}$$

$$:= 0 \text{ otherwise} \tag{58}$$

and similarly for $\bar{u}^L_\kappa(x)$.

Unruh (1976) used the analytic properties of the eigenfunctions $u^R_\kappa(x)$ to find the Bogoliubov transformation between the above expansion and the usual one:

$$\phi(x) = \int \frac{d^3\kappa}{(2\omega_\kappa)^{1/2}} \left(a(\mathbf{k})\, e^{ik\cdot x} + a^\dagger(\mathbf{k})\, e^{-ik\cdot x}\right) \tag{59}$$

Unruh showed that the inertial vacuum can be written as a thermal density matrix in the Fock basis associated with the accelerating observer. It is this result that leads to the claim that an accelerating observer appears to be immersed in a thermal bath.

3.3. The Histories Approach

We now formulate QFT on Rindler space–time using the HPO approach and show how the the Unruh effect appears within the formalism.

First we use the time coordinate of our accelerating observer to foliate $\mathcal{R}$ with a one-parameter family of spacelike Cauchy surfaces $\mathcal{C}^{\mathcal{R}}_\tau$ where

$$\mathcal{C}^{\mathcal{R}}_\tau := \{r \in \mathcal{R} \mid \xi^0(r) = \tau\} \tag{60}$$

The single time Hilbert space for the theory is then $\mathcal{H}_\tau := \mathcal{F}[L^2(\mathcal{C}^{\mathcal{R}}_\tau, d\mu]$ where $d\mu(\xi) = (\alpha\xi^1)^{-1} d^3\xi$ (Fulling, 1973). The History space is

$$\mathcal{V}^{\mathcal{R}} := \otimes_{\tau\in\mathbb{R}} \mathcal{H}_\tau \sim \mathcal{F}[L^2(\mathcal{R}, d\mu\, d\tau)] \tag{61}$$

By analogy with Eq. (39) we define the history algebra to be

$$[\phi_{\tau_1}(\vec{\xi}_1), \phi_{\tau_2}(\vec{\xi}_2)] = 0 \tag{62}$$

$$[\pi_{\tau_1}(\vec{\xi}_1), \pi_{\tau_2}(\vec{\xi}_2)] = 0 \tag{63}$$

$$[\phi_{\tau_1}(\vec{\xi}_1), \pi_{\tau_2}(\vec{\xi}_2)] = i\hbar\delta(\tau_1 - \tau_2)\delta^3(\vec{\xi}_1 - \vec{\xi}_2) \tag{64}$$

with $\vec{\xi}_1 \in \mathcal{C}_{\tau_1}$.

The Hamiltonian of the real scalar field in $\mathcal{R}$ is

$$H^R_\tau = \frac{1}{2}\int d^3\xi\alpha\xi^1\left(\pi^R_\tau(\vec{\xi})^2 + \nabla_\xi\phi^R_\tau(\vec{\xi})\cdot\nabla_\xi\phi^R_\tau(\vec{\xi}) + m^2\phi^R_\tau(\vec{\xi})^2\right) \tag{65}$$

where the vector field ∇_ξ is defined by $\nabla_\xi := \partial_{\xi^1} + \partial_{\xi^2} + \partial_{\xi^3}$, and the dot product is taken using the 3-metric on $\mathcal{C}^{\mathcal{R}}_\tau$; $g^3 = \text{diag}(-1, -1, -1)$. Equation (65) has the same form for all τ so the representation of the history algebra in which H^R_τ is self-adjoint is isomorphic on each $\mathcal{H}_\tau$. The commutation relations of the smeared Hamiltonian with $\phi^R_\tau(\vec{\xi})$ and $\pi^R_\tau(\vec{\xi})$ are

$$\left[H^R_f, \phi^R_\tau(\vec{\xi})\right] = -i\hbar\alpha\xi^1 f(\tau)\pi^R_\tau(\vec{\xi}) \tag{66}$$

$$\left[H^R_f, \pi^R_\tau(\vec{\xi})\right] = i\hbar f(\tau)K_R\phi^R_\tau(\vec{\xi}) \tag{67}$$

where K_R is defined by $(K_R f)(\tau, \vec{\xi}) := (-\nabla_\xi(\alpha\xi^1\nabla_\xi) + \alpha\xi^1 m^2) f(\tau, \vec{\xi})$. Now we can follow the analysis of Isham *et al.* (1998) to show that there is a unitary representation of the exponentiated commutation relations and that therefore the Hamiltonian exists as a self-adjoint operator in this representation. We can deduce the associated annihilation operators to be

$$b^R_\tau(\vec{\xi}) = \frac{1}{\sqrt{2}}\left(K_R^{1/4}\phi^R_\tau(\vec{\xi}) + i\frac{\alpha\xi^1}{K_R^{1/4}}\pi^R_\tau(\vec{\xi})\right) \tag{68}$$

This defines a particular complexification of the test function space that is equivalent to a choice of positive and negative frequencies consistent with the Killing field ∂_{ξ^0}. Using these creation and annihilation operators we can build the Fock basis for the history theory. These equations can be written in a more familiar form by taking the spectral transform of the $b^R_\tau(\vec{\xi})$ and $b^{R^\dagger}_\tau(\vec{\xi})$, that is by expanding them in terms of the eigenfunctions of K_R, $u^R_\kappa(\vec{\xi})$[2]:

$$\phi^R_\tau(\vec{\xi}) := \int \frac{d^3k}{(2\omega_\kappa)^{1/2}}\left(u^R_\kappa(\vec{\xi})\, b^{R^\dagger}_\tau(\vec{\kappa}) + u^R_\kappa(\vec{\xi}) b^R_\tau(\vec{\kappa})\right) \tag{69}$$

as before. There is obviously a strong similarity between the histories version of this problem and the canonical version. But, from the histories perspective the result of Unruh shows nothing because a thermal density matrix is not a projection operator and so has no meaning when defined on $\mathcal{V}^{\mathcal{M}}$. Only elements of $\mathcal{P}(\mathcal{V}^{\mathcal{M}})$ and $\mathcal{P}(\mathcal{V}^{\mathcal{R}})$ are meaningful in a history theory as these can be considered as propositions about histories, that is, as elements of $\mathcal{UP}^{\mathcal{M}}$ and $\mathcal{UP}^{\mathcal{R}}$. We have to change our approach so that we are talking about projectors onto eigenvectors of the average Rindler particle number operator:

$$N_\kappa := \int d\tau\, b^{R^\dagger}_\tau(\vec{\kappa}) b^R_\tau(\vec{\kappa}) \tag{70}$$

These vectors are of the form

$$\left|n^\kappa_f\right\rangle := (n!)^{-1/2}\int d\tau\ f(\tau)(b^{R^\dagger}_\tau(\vec{\kappa}))^n|0^R\rangle \tag{71}$$

and have a degenerate spectrum in the same way as those for the inertial observer because we are still considering a free theory. Projectors onto these vectors

[2] These are just the function $u^R_\kappa(\xi)$, but with the time dependent part set to 1.

represent propositions about the time-averaged number of particles in each mode, as seen by the accelerating observer.

The space of propositions about possible histories is not the same for the accelerating observer as for the inertial observer, but this is not the only difference. The decoherence functional associated with a quantum system depends on both the initial conditions and the Hamiltonian. The accelerating observer has a different Hamiltonian to the inertial observer and so has a different decoherence functional.

As before, the fact that $[N_\kappa, H^R] = 0$ implies that

$$d^{\mathcal{R}}(m_f^\kappa, n_g^{\kappa'}) = \delta_{mn}\delta^3(\kappa - \kappa')\rho^R_{m^\kappa n^\kappa} \tag{72}$$

for all f, g in notation which parallels that of (49) but now $\rho^R \in \mathcal{B}(\mathcal{H}^R_{\tau 0})$ and $|n^\kappa\rangle \in \mathcal{H}^R_{\tau 0}$ is defined by

$$|n^\kappa\rangle := \left(b^{R^\dagger}_{\tau_0}(\kappa)\right)^n |0^R\rangle \tag{73}$$

We can write this in the form,

$$d^{\mathcal{R}}\left(m_f^\kappa, n_g^\kappa\right) := \mathrm{Tr}_{\mathcal{V}^{\mathcal{R}}\otimes\mathcal{V}^{\mathcal{R}}}\left[P_{m_f^\kappa} \otimes P_{n_g^\kappa}\mathcal{Z}^{\mathcal{R}}\right] \tag{74}$$

for $\mathcal{Z}^{\mathcal{R}} \in \mathcal{B}(\mathcal{V}^{\mathcal{R}} \otimes \mathcal{V}^{\mathcal{R}})$ defined similarly to the Minkowski case, (52).

3.4. The Unruh Effect

Finally we can see how the Unruh effect arises in the HPO formalism. Let us consider the situation in the inertial vacuum, that is, the initial density matrix is

$$\rho^M_{n^{\mathbf{k}} n^{\mathbf{k}}} = \delta_{0n} \tag{75}$$

for all $\mathbf{k} \in \mathbb{R}^3$, where the matrix elements are taken in the Fock representation associated with the inertial observer. Note that this density matrix means that the probability of the inertial observer detecting n particles in any mode is zero unless $n = 0$:

$$d^{\mathcal{M}}\left(m_f^{\mathbf{k}}, n_g^{\mathbf{k}'}\right) = \delta_{mn}\delta(\mathbf{k} - \mathbf{k}')\delta_{0n} \tag{76}$$

The density matrix ρ^M is defined on some initial Hilbert space $\mathcal{H}_{t_0}$, but we can choose our Cauchy surfaces so that

$$\mathcal{H}_{t_0} = \mathcal{H}^{\mathcal{L}}_{\tau_0} \otimes \mathcal{H}^{\mathcal{R}}_{\tau_0} \tag{77}$$

Using Unruh's result on this initial Hilbert space we can write the inertial vacuum as a thermal density matrix in the representation associated with the accelerating observer. Tracing over $\mathcal{H}^{\mathcal{L}}_{\tau_0}$ we obtain the initial condition for the accelerating observer (Unruh and Wald, 1984)

$$\rho^R_{n^k n^k} = N_{\frac{2\pi}{\alpha}}(n\omega_\kappa) \tag{78}$$

where $N_\beta(E)$ is the thermal distribution giving the probability of a scalar particle having energy E in a heat bath of inverse temperature β. Finally,

$$d^{\mathcal{R}}\left(n_f^\kappa, m_g^{\kappa'}\right) = \delta_{mn}\delta(\kappa - \kappa')N_{\frac{2\pi}{\alpha}}(n\omega_\kappa) \tag{79}$$

which shows that the accelerating observer detects a thermal spectrum at inverse temperature $\beta = \frac{2\pi}{\alpha}$, in agreement with the result of Unruh.

4. CONCLUSION

We have shown that it is possible to consider average number propositions within the continuous-time HPO formalism. We have postulated a condition on the decoherence functional which ensures that energy propositions form a consistent set, as they do in the conventional theory, and which gives the correct probabilities for such propositions. This condition is defined for the SHO and the QFT but can easily be generalised to any system with symmetries because its construction involves only the matrix elements of the initial density matrix in the basis associated with the symmetry.

We have shown that the HPO scheme allows the construction of QFT in curved space–time and have rederived the well-known result of Unruh within this scheme. It is a straightforward matter to extend this method to the case of Hawking radiation and, more generally, to any of the thermal gravitational effects discussed in the literature. In fact, the general nature of the HPO formalism—in particular its ability to cope with very general temporal support strucures and the associated nonunitary evolution—means that it can potentially be used to formulate QFT on much more general space–times such as nonglobally hyperbolic space–times or those with topology change. This remains a task for future research.

Another potentially interesting avenue of research is to attempt to apply the formalism to other problems in conventional QFT such as scattering. Scattering type questions typically involve propositions such as "there are n_1 particles of type 1 at time t_1 and then n_2 particles of type 2 at time t_2." We cannot pose such questions in the formalism as presented here because we cannot embed discrete time propositions into the continuous-time history space. The best we can do is to use propositions with support in a neighborhood of t_1 and t_2, which we can arbitrarily choose to be small. Nontrivial scattering questions necessarily involve interactions and we haven't considered these here, but in principle there is no reason why perturbation theory could not be developed.

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